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INTERLACING PROPERTIES OF THE ZEROS OF THE ERROR FUNCTION IN BE--ETC(U)
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INTERLACING PROPERTIES OF THE ZEROS OF THE ERROR FUNCTION IN BEST L^p-APPROXIMATION, $1 \le p \le \infty$

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ABSTRACT

Let $\{u_i\}_{i=1}^n$, φ, ψ be functions in $C(\bar{I})$, where I is a finite interval. Let $d\sigma$ be a finite, non-atomic, positive measure on I. For $p \in [1,\infty]$, let $E_p(\varphi)$ and $E_p(\psi)$ denote the error functions in the best L^p-approximation to φ and ψ, respectively, from $[u_1, \ldots, u_n]$. For $p < \infty$, the L^p-approximation is taken with respect to $d\sigma$.

Theorem. Assume $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ are Tchebycheff systems on I. Then for $1 < p < \infty$, the zeros of $E_p(\varphi)$ and $E_p(\psi)$ in I strictly interlace. For $p = 1$, either the zeros strictly interlace or $E_1(\varphi)$ has exactly n sign changes and $\operatorname{sgn} E_1(\varphi)(t) = \operatorname{sgn} E_1(\psi)(t)$ for all $t \in \operatorname{int}(I)$. For $p = \infty$, the strict interlacing is present when I is a closed interval.

Applications to questions of interlacing of zeros in polynomial distortion problems are given.

AMS(MOS) Subject Classification - 41A50

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INTERLACING PROPERTIES OF THE ZEROS OF THE ERROR FUNCTION IN BEST L^p-APPROXIMATION, $1 \le p \le \infty$

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1. Introduction

Let $\{u_i\}_{i=1}^n, \varphi, \psi$ be functions in $C(\bar{I})$, where I is a finite interval. We shall henceforth assume that $\bar{I} = [0,1]$. Let $d\sigma$ be a finite, non-atomic, positive measure on I. For $p \in [1,\infty]$, let $E_p(\varphi)$ and $E_p(\psi)$ denote the error functions in the best L^p-approximation to φ and ψ, respectively, from $[u_1,\ldots,u_n]$. For $p < \infty$, the L^p-approximation is taken with respect to $d\sigma$.

The main result of this paper, whose proof will be carried out separately for $1 < p < \infty$, $p = 1$, and $p = \infty$ in Sections 3-5, is:

Theorem 1.1. *Assume* $\{u_1,\ldots,u_n\}$ *and* $\{u_1,\ldots,u_n,\varphi,\psi\}$ *are Tchebycheff (T) systems on* I, $n \ge 1$. *Then, for* $1 < p < \infty$, *the zeros of* $E_p(\varphi)$ *and* $E_p(\psi)$ *in* I *strictly interlace. For* $p = 1$, *either the zeros strictly interlace, or* $E_1(\varphi)$ *has exactly* n *sign changes and* $\operatorname{sgn} E_1(\varphi)(t) = \operatorname{sgn} E_1(\psi)(t)$ *for all* $t \in \operatorname{int}(I)$. *For* $p = \infty$, *strict interlacing is present when* I *is a closed interval.*

For the case $p = 1$, the interlacing holds under somewhat less restrictive conditions on φ and ψ (see Theorem 4.2). On the other hand, for $p = \infty$, interlacing properties for the points of alternation are also available (see Theorem 5.2).

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Assume $\{u_1, \ldots, u_k\}$ is a T-system on I, for $k = n, n+1, n+2$.
Let $q_{k,p}$, $k = n, n+1$ be the error function in the best L^p-approximation of u_{k+1} from $[u_1, \ldots, u_k]$. If $q_{n+1,p} = u_{n+2} - \sum_{i=1}^{n+1} \alpha^*_{i,p} u_i$, then $q_{n+1,p} = E_p(u_{n+2} - \alpha^*_{n+1,p} u_{n+1})$ where E_p is as previously defined. We observe next that $\varphi = u_{n+1}$ and $\psi = u_{n+2} - \alpha^*_{n+1,p} u_{n+1}$ satisfy the conditions of Theorem 1.1 and that $q_{k,p}$ has exactly k sign changes in int I, $k = n, n+1$. Hence, we now have

Theorem 1.2. The n zeros of $q_{n,p}$ and the n+1 zeros of $q_{n+1,p}$ strictly interlace in I for $1 \leq p < \infty$. If $I = \bar{I}$, the interlacing holds for $p = \infty$ as well.

When $\{u_i\}_{i=1}^{n+2}$ are the monomials $\{x^{i-1}\}_{i=1}^{n+2}$, the interlacing properties for $p = 2$ are a well known consequence of the fact the $q_{k,2}$ are the orthogonal polynomials on I with respect to the measure $d\sigma$. For $p = 1$ and $d\sigma(t) = dt$, the Lebesgue measure, $q_{k,1}$ is the $k^{\underline{th}}$ Tchebycheff polynomial of the second kind, and the interlacing follows from the explicit expression for $q_{k,1}$. For $p = \infty$, $q_{k,\infty}$ is the $k^{\underline{th}}$ Tchebycheff polynomial of the first kind and the interlacing is known. Somewhat more general results for $p = \infty$, where $u_i(t) = t^{i-1}$, $i = 1, \ldots, n$, have been discussed in the literature (see Paszkowski [5], Rowland [6], and Shohat [7]). For $p \neq 1, 2, \infty$, the results are new even for the monomial case. For $p = 2$, a known result on the interlacing of roots of quasi-orthogonal polynomials (see [9, p. 5-10]) is subsumed by Theorem 1.1.

Some recent applications of the present results serve to establish the interlacing of the zeros in (0, 1) of $P_{n,p}$ and $P_{n+1,p}$, where $P_{n,p}$

is the unique solution of

$$\min\{ \|P_n\|_p : P_n \in T_n, \ \|P_n\|_\infty = 1\}$$

normalized so that $P_{n,p}(0) = 1$, and T_n is the set of all trigonometric polynomials of degree $\leq n$ (see [8]). An essentially similar property is available for the case where T_n is replaced by π_n, the set of algebraic polynomials of degree $\leq n$ (see [1]).

2. Preliminaries

Let I and $d\sigma$ be as previously indicated. In this section we recall some basic facts concerning continuous T-systems on I. These facts, with perhaps minor modifications, may all be found in Karlin and Studden [3], and Gantmacher and Krein [2].

Definition 2.1. The system $\{u_i\}_{i=1}^n$ of continuous functions on an interval I is called a T-system if every non-trivial linear combination of $\{u_i\}_{i=1}^n$ has at most $n-1$ zeros in I.

The following concepts will prove relevant.

Definition 2.2. For any $f \in C(I)$, we call $t_0 \in \mathrm{int}(I)$ a non-nodal zero of f provided that f vanishes, but does not change sign at t_0. All other zeros are called nodal.

Definition 2.3. For $f \in C(I)$, let $Z(f)$ denote the number of zeros of f in I, with the convention that non-nodal zeros are counted twice.

Definition 2.4. We say that a function f defined on $I = [a, b]$ has k sign changes

on I if I is decomposable into $k+1$ intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, k+1$, $x_0 = a < x_1 < \ldots < x_{k+1} = b$, such that $\epsilon(-1)^i f(t) \geq 0$ for $t \in I_i$, $i = 1, \ldots, k+1$, $\varepsilon = +1$ or -1 fixed, and f does not vanish identically on I_i. We denote the number of sign changes of f on I by $S^-(f)$.

Lemma 2.1. $\{u_i\}_{i=1}^n$ is a T-system on I iff $Z(u) \leq n-1$ for any $u(t) = \sum_{i=1}^n a_i u_i(t)$, $\sum_{i=1}^n a_i^2 > 0$.

Lemma 2.2. Let $\{u_i\}_{i=1}^n$ be a T-system on I. For any k prescribed distinct points in $\operatorname{int}(I)$, $k \leq n-1$, there exists a $u(t) = \sum_{i=1}^n a_i u_i(t)$ with nodal zeros at these points, and which vanishes nowhere else in $\operatorname{int}(I)$.

With the aid of the above lemma, it is a simple matter to prove the following result.

Lemma 2.3. Let $\{u_i\}_{i=1}^n$ be a T-system on I, $u_i \in C(\bar{I})$, $i = 1, \ldots, n$. If $v(t)$ is a bounded function on $\bar{I}$, with at most a finite number of discontinuities and

$$\int_I v(t)\, u_i(t)\, d\sigma(t) = 0, \qquad i = 1, \ldots, n,$$

then $S^-(v) \geq n$.

3. $1 < p < \infty$

Let $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ be T-systems on I, and assume $\{u_i\}_{i=1}^n, \varphi, \psi \in C(\bar{I})$. For fixed $p \in (1, \infty)$, let $g_1(t) = E_p(\varphi)(t)$ and $g_2(t) = E_p(\psi)(t)$, where $E_p(\varphi)$, $E_p(\psi)$ are as defined in the introduction.

Theorem 3.1. The zeros of $g_1(t)$ and $g_2(t)$ in I strictly interlace.

The proof of Theorem 3.1 is divided into a series of lemmas and propositions. In the first part of this section, we prove the following result.

Proposition 3.1. For $g_1(t)$ and $g_2(t)$ as above,

$$n \leq S^-(\alpha g_1 + \beta g_2) \leq Z(\alpha g_1 + \beta g_2) \leq n+1$$

for all α, β real, $\alpha^2 + \beta^2 > 0$.

Since $\{u_1, \ldots, u_n, \varphi, \psi\}$ is a T-system on I, we immediately obtain from Lemma 2.1,

Lemma 3.1. For $g_1(t)$ and $g_2(t)$ as above,

$$Z(\alpha g_1 + \beta g_2) \leq n+1 \quad ,$$

for all α, β real, $\alpha^2 + \beta^2 > 0$.

Set

$$\begin{aligned} h_1(t) &= [\operatorname{sgn} g_1(t)]\, |g_1(t)|^{p-1} \\ h_2(t) &= [\operatorname{sgn} g_2(t)]\, |g_2(t)|^{p-1} \quad . \end{aligned} \tag{3.1}$$

From the orthogonality relations characterizing the unique best L^p-approximation on I from $\{u_i\}_{i=1}^{n}$, it follows that

$$\int_I h_j(t)\, u_i(t)\, d\sigma(t) = 0, \qquad i = 1, \ldots, n; \; j = 1, 2 \quad . \tag{3.2}$$

A direct application of Lemma 2.3 and (3.2) yields:

Lemma 3.2. For all α, β real, $\alpha^2 + \beta^2 > 0$,

$$S^-(\alpha h_1 + \beta h_2) \geq n \quad .$$

Remark 3.1. For $p = 2$, Proposition 3.1 is an immediate result of Lemmas 3.1 and 3.2 since in this case $h_i(t) = g_i(t)$, $i = 1, 2$.

From (3.1), $h_i(t)$ and $g_i(t)$ obviously have the same zero properties, with respect to both S^- and Z . Thus, from Lemma 3.2,

Lemma 3.3. For $g_1(t)$ and $g_2(t)$ as above,

$$n \leq S^-(g_i), \qquad i = 1, 2 \quad .$$

Therefore, in the proof of Proposition 3.1 we shall assume, without loss of generality, that $\alpha = 1$, $\beta \neq 0$.

Lemma 3.4. For $h_i(t)$ and $g_i(t)$ as above, $i = 1, 2$, and $t_0 \in I$, $g_1(t_0) + \beta g_2(t_0) = 0$ iff $h_1(t_0) + [\operatorname{sgn} \beta] |\beta|^{p-1} h_2(t_0) = 0$.

Proof. We disregard the case where $h_i(t_0)$ or $g_i(t_0)$ is zero for some i, since any one of the four terms being zero implies, since $\beta \neq 0$, that all the remaining terms are zero, and the lemma is proven.

Assume $g_1(t_0) + \beta g_2(t_0) = 0$. Thus,

$$|g_1(t_0)| = |\beta| |g_2(t_0)|, \text{ and } [\operatorname{sgn} g_1(t_0)] = -[\operatorname{sgn} \beta] [\operatorname{sgn} g_2(t_0)] \quad .$$

Therefore,

$$\begin{aligned} h_1(t_0) &= [\operatorname{sgn} g_1(t_0)] \ |g_1(t_0)|^{p-1} \\ &= -[\operatorname{sgn} \beta] [\operatorname{sgn} g_2(t_0)] \ |\beta|^{p-1} |g_2(t_0)|^{p-1} \\ &= -[\operatorname{sgn} \beta] \ |\beta|^{p-1} \ h_2(t_0) \quad . \end{aligned}$$

The above analysis is totally reversible and the lemma is proven.

Lemma 3.5. $g_1(t) + \beta g_2(t)$ has a non-nodal zero at $t_0 \in \text{int}(I)$ iff $h_1(t) + [\text{sgn}\,\beta]\,|\beta|^{p-1}\,h_2(t)$ has a non-nodal zero at t_0.

Proof. Assume, without loss of generality, that $g_1(t) + \beta g_2(t) \geq 0$ for t in some small neighborhood of t_0, with equality holding for $t = t_0$.

Case 1. $g_1(t_0) \neq 0$.

Since $g_1(t_0) \neq 0$, we have $g_2(t_0) \neq 0$. Let N denote a neighborhood of t_0 in which $g_1(t)$, $g_2(t)$ are never zero, and $g_1(t) + \beta g_2(t) \geq 0$.

Since equality holds for $t = t_0$, we have

$$[\text{sgn}\, g_1(t)] = -[\text{sgn}\,\beta]\,[\text{sgn}\, g_2(t)], \quad \text{for } t \in N. \tag{3.3}$$

If $g_1(t_0) > 0$, then, for $t \in N$

$$g_1(t) = |g_1(t)| \geq |\beta|\,|g_2(t)|. \tag{3.4}$$

Raising both sides to the $p-1$ st power, and multiplying by the expressions in (3.3), which are positive in this case, we obtain the desired result.

If $g_1(t_0) < 0$, then (3.4) is reversed, but the expressions in (3.3) are negative. The result once again follows.

Case 2. $g_1(t_0) = 0$.

Thus $g_2(t_0) = h_2(t_0) = h_1(t_0) = 0$. Since $g_i(t)$ and $h_i(t)$ exhibit the same zero structure, it follows from Lemmas 3.1 and 3.3 that $g_i(t)$ and $h_i(t)$ necessarily change sign at t_0 for $i = 1, 2$.

Let N be a sufficiently small neighborhood of t_0 such that the only zero of $g_i(t)$, $h_i(t)$, $i = 1, 2$, and $g_1(t) + \beta g_2(t)$ in N is at t_0.

Assume $g_1(t) > 0$ for $t > t_0$, $t \in N$. Since $|g_1(t)| \leq \beta g_2(t)$ for $t \leq t_0$, $t \in N$, we have $[\operatorname{sgn} \beta] = [\operatorname{sgn} g_2(t)]$ for $t < t_0$, $t \in N$. Since both $g_1(t)$ and $g_2(t)$ change sign at t_0, $|g_1(t)|^{p-1} \geq |\beta|^{p-1} |g_2(t)|^{p-1}$ for $t \geq t_0$, $t \in N$, and $|g_1(t)|^{p-1} \leq |\beta|^{p-1} |g_2(t)|^{p-1}$ for $t \leq t_0$, $t \in N$. The result now follows from the definitions of $h_1(t)$ and $h_2(t)$. The case $g_1(t) < 0$ for $t > t_0$, $t \in N$, is totally analogous.

Since the analysis of Cases 1 and 2 is reversible, the lemma is proven.

Proof of Proposition 3.1. Apply Lemmas 3.1-3.5.

The next proposition is a modification of a result of Gantmacher and Krein [2], (see also Lee and Pinkus [4]).

Proposition 3.2. *If $\Phi, \Psi \in C(0,1)$, and $n \leq S^-(\alpha\Phi + \beta\Psi) \leq Z(\alpha\Phi + \beta\Psi) \leq n+1$ for all real α, β, $\alpha^2 + \beta^2 > 0$, then the zeros of Φ and Ψ in $(0,1)$ strictly interlace.*

The proof of Proposition 3.2 is also divided into a series of lemmas. Note the important fact that the above inequalities imply that $\alpha\Phi(t) + \beta\Psi(t)$ has no non-nodal zeros in $(0,1)$.

Let $\{\xi_i\}_{i=1}^k$, $\xi_0 = 0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = 1$, $(k = n$ or $n+1)$ denote the zeros (sign changes) of $\Phi(t)$ in $(0,1)$. Let $I_i = (\xi_{i-1}, \xi_i)$, $i = 1, \ldots, k+1$, and $f(t) = \dfrac{\Psi(t)}{\Phi(t)}$.

Lemma 3.6. $f(t)$ is strictly monotone in each I_i, $i = 1,\ldots,k+1$.

Proof. If $f(t)$ is a constant c on any interval of I_i of positive length, then $Z(\Psi - c\Phi) = \infty$, contradicting the hypothesis of the proposition. If f is not strictly monotone on I_i, then f has a relative extremum at some point $x_i \in I_i$. The function $\Psi(t) - f(x_i)\Phi(t)$ has a non-nodal zero at x_i, contradicting the hypothesis. The lemma is proven.

Lemma 3.7. $f(t)$ has exactly one zero in each I_i, $i = 2,\ldots,k$.

Proof. Since $f(t)$ is monotone in each I_i, $i = 1,\ldots,k+1$, the limits

$$\lim_{t\to\xi_i^-} f(t) = \ell_i^- \quad \text{and} \quad \lim_{t\to\xi_i^+} f(t) = \ell_i^+$$

both exist as extended real numbers for $i = 1,\ldots,k$. We shall show that none of the $\{\ell_i^+\}_{i=1}^k$ and $\{\ell_i^-\}_{i=1}^k$ is finite. Taken together with Lemma 3.6, this implies the statement of the lemma.

Let us assume that either ℓ_i^- or ℓ_i^+ is finite. Since $\Phi(\xi_i) = 0$, it follows that $\Psi(\xi_i) = 0$. We are concerned with one of the following four cases.

(i) exactly one of ℓ_i^+ and ℓ_i^- is finite,

(ii) ℓ_i^+ and ℓ_i^- are finite and unequal

(iii) $\ell_i^+ = \ell_i^-$ (finite) and f is monotone in a neighborhood of ξ_i.

(iv) $\ell_i^+ = \ell_i^-$ (finite) and f is monotone in opposite senses for $t \in I_i$ and $t \in I_{i+1}$.

If either cases (i) or (ii) occur, let c be any real number between

ℓ_i^+ and ℓ_i^-, while if case (iii) holds, let $c = \ell_i^+ = \ell_i^-$. Then $\Psi(t) - c\Phi(t)$ has a non-nodal zero at ξ_i since $\Phi(\xi_i) = \Psi(\xi_i) = 0$, and $\Phi(t)$ changes sign at ξ_i.

Assume case (iv) obtains. Let $c = \ell_i^+ = \ell_i^-$ and assume, without loss of generality, that $f(t) \leq c$ for t in a neighborhood of ξ_i. Now, $\Psi(t) - c\Phi(t)$ has at least n sign changes in $(0,1)$, one of which is at ξ_i. Thus $\Psi(t) - c\Phi(t) + \epsilon\Phi(t)$ has for $\epsilon > 0$, ϵ sufficiently small, at least $n-1$ sign changes bounded away from ξ_i. Since $f(t)$ is strictly monotone in I_i and I_{i+1}, $\Psi(t) - (c-\epsilon)\Phi(t)$ has a zero slightly to the left of ξ_i, a zero slightly to the right of ξ_i, and vanishes at ξ_i. Thus $\Psi(t) - (c-\epsilon)\Phi(t)$ has at least $n+2$ zeros in $(0,1)$. A contradiction. The lemma is proven.

Since $\Phi(t)$ and $\Psi(t)$ are interchangeable in the above analysis, Proposition 3.2, for $n \geq 2$, follows from Lemmas 3.6 and 3.7. For the cases $n = 0$ and $n = 1$, the following lemma is also used.

Lemma 3.8. $\Phi(t)$ and $\Psi(t)$ have no common zeros in $(0,1)$.

Proof. Assume $\Phi(\xi) = \Psi(\xi) = 0$. Let $f(t) = \frac{\Psi(t)}{\Phi(t)}$ and $g(t) = \frac{\Phi(t)}{\Psi(t)}$. Both $f(t)$ and $g(t)$ are, by Lemma 3.6, strictly monotone in some neighborhood to the left and in some neighborhood to the right of ξ. Furthermore, the limits as $t \to \xi$, from above and below, exist and are infinite by Lemma 3.7. However, both $\Phi(t)$ and $\Psi(t)$ change sign at ξ and a contradiction immediately ensues. The lemma is proven.

The proof of Proposition 3.2 is complete.

Proof of Theorem 3.1. If I is an open interval, then Theorem 3.1 is a consequence of Propositions 3.1 and 3.2.

Assume $I = [0,1)$ and $g_1(0) = 0$. Since $n \geq 1$, let $\xi \in (0,1)$ be such that $g_1(\xi) = 0$ and $g_1(t) \neq 0$ for all $t \in (0,\xi)$. From Proposition 3.2, $g_2(\xi) \neq 0$. We must prove that $g_2(0) \neq 0$ and $g_2(t)$ has a zero in $(0,\xi)$. Assume $g_2(t)$ has no zero in $[0,\xi]$. This immediately contradicts the monotonicity of $\frac{g_2(t)}{g_1(t)}$ in $(0,\xi)$ (see Lemma 3.6). Now, assume $g_2(0) = 0$, and by interchanging $g_1(t)$ and $g_2(t)$, if necessary, assume $g_2(t) \neq 0$ in $(0,\xi]$. Assume also that $g_1(t)\, g_2(t) > 0$ for $t \in (0,\xi)$. Therefore $\lim\limits_{t \to \xi -} \frac{g_2(t)}{g_1(t)} = \infty$ and $\lim\limits_{t \to 0+} \frac{g_2(t)}{g_1(t)} \downarrow c \geq 0$, c finite. $g_2(t) - cg_1(t)$ has n sign changes in $(0,1)$ and thus for $\epsilon > 0$, ϵ sufficiently small, $g_2(t) - (c+\epsilon)\, g_1(t)$ has n sign changes in $(0,1)$ bounded away from $t = 0$, a zero near $t = 0$, and a zero at $t = 0$. Therefore $g_2(t) - (c+\epsilon)\, g_1(t)$ has at least $n+2$ zeros in $I = [0,1)$, a contradiction.

This same analysis applies where $I = (0,1]$ and $I = [0,1]$. The theorem is proven.

4. $\underline{p = 1}$

As previously, let $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ be T-systems on I, and assume $\{u_i\}_{i=1}^{n}, \varphi, \psi \in C(\bar{I})$. Let $g_1(t) = E_1(\varphi)(t)$ and $g_2(t) = E_1(\psi)(t)$, where $E_1(\varphi)$ and $E_1(\psi)$ are as defined in the introduction.

In this section we prove the following result.

<u>Theorem 4.1.</u> <u>The zeros of</u> $g_1(t)$ <u>and</u> $g_2(t)$ <u>on</u> I <u>strictly interlace unless</u> $S^-(g_1) = S^-(g_2) = n$, <u>in which case</u> $[\operatorname{sgn} g_1(t)] = [\operatorname{sgn} g_2(t)]$ <u>for all</u> $t \in \operatorname{int}(I)$.

We define $h_j(t)$, $j = 1,2$ on $\bar{I}$ as follows. Set $h_j(t) = \operatorname{sgn} g_j(t)$ for $t \in \operatorname{int}(I)$, and let $h_j(t)$ be continuous at the endpoints, $j = 1,2$. Since $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ are T-systems on I, $g_1(t)$ and $g_2(t)$ are uniquely defined, and since $Z(g_j) \le n+1$, $j = 1,2$, $|h_j(t)| = 1$ a.e. on I, $j = 1,2$, and satisfy the orthogonality relations

$$\int_I h_j(t)\, u_i(t)\, d\sigma(t) = 0, \qquad i = 1, \ldots, n;\ j = 1,2 . \tag{4.1}$$

Lemma 4.1. *For $h_1(t)$ and $h_2(t)$ as above,* $n \le S^-(h_j) \le n+1$, $j = 1,2$, *and* $n \le S^-(h_1 \pm h_2)$, *unless* $h_1(t) \pm h_2(t) \equiv 0$ *on* I .

Proof. This is an immediate consequence of Definition 2.1 and Lemma 2.3.

Replacing $h_j(t)$ by $-h_j(t)$, if necessary and letting $\bar{I} = [0,1]$, we may assume the existence of $\{\xi_i\}_{i=1}^k$ and $\{\eta_i\}_{i=1}^m$, $n \le k, m \le n+1$, where

$$\xi_0 = 0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = 1$$

$$\eta_0 = 0 < \eta_1 < \ldots < \eta_m < \eta_{m+1} = 1$$

such that

$$\begin{aligned} h_1(t) &= (-1)^i, \quad \xi_i < t < \xi_{i+1}, \qquad i = 0,1,\ldots,k \\ h_2(t) &= (-1)^i, \quad \eta_i < t < \eta_{i+1}, \qquad i = 0,1,\ldots,m . \end{aligned} \tag{4.2}$$

Lemma 4.2. *For $h_1(t)$ and $h_2(t)$ as above,* $S^-(h_1 \pm h_2) \le \min\{k,m\}$, *and if* $k = m$, *then* $S^-(h_1 - h_2) \le k-1 = m-1$.

Proof. The above lemma is known. For completeness, we include a proof. With no loss of generality, assume $k \le m$. From the definition of $h_1(t)$,

$$(h_1(t) \pm h_2(t))\,(-1)^i \geq 0, \quad \xi_i < t < \xi_{i+1}, \quad i = 0,1,\ldots,k .$$

Thus $S^-(h_1 \pm h_2) \leq k = \min\{k, m\}$.

Assume $k = m$ and $\xi_1 \leq \eta_1$. Since $h_1(t) - h_2(t) \equiv 0$ on $[0, \xi_1)$, $S^-_{(0,1)}(h_1 - h_2) = S^-_{(\xi_1,1)}(h_1 - h_2)$. However $h_1(t)$ has $k-1$ sign changes on $(\xi_1, 1)$. Applying the previous result, the lemma is proven.

<u>Lemma 4.3.</u> <u>If</u> $S^-(g_1) = S^-(g_2) = n$, <u>then</u> $\xi_i = \eta_i$, $i = 1,\ldots,n$.

<u>Proof</u>. Since $S^-(n_j) = S^-(g_j)$, $j = 1,2$, then $S^-(h_1 - h_2) \leq n-1$ by Lemma 4.2. From Lemma 4.1, it follows that $h_1(t) \equiv h_2(t)$ for almost all $t \in [0,1]$. Thus $\xi_i = \eta_i$, $i = 1,\ldots,n$, and the lemma is proven.

The above lemma is a restatement of the well-known fact that if $\{u_1,\ldots,u_n\}$ is a T-system on $(0,1)$, then there exist n unique points $\{\zeta_j\}_{j=1}^n$, $\zeta_0 = 0 < \zeta_1 < \ldots < \zeta_n < \zeta_{n+1} = 1$, such that

$$\sum_{j=0}^{n} (-1)^j \int_{\zeta_j}^{\zeta_{j+1}} u_i(t)\, d\sigma(t) = 0, \qquad i = 1,\ldots,n .$$

To prove Theorem 4.1, it remains to consider the case where at least one of $S^-(h_1)$, $S^-(h_2)$ is $n+1$. Note that if $S^-(h_j) = S^-(g_j) = n+1$, $j = 1,2$, then we cannot have $h_1(t) \equiv h_2(t)$ for almost all $t \in I$. This is a consequence of the fact that there exists a unique (up to a multiplicative constant) non-trivial linear combination of $\{u_1,\ldots,u_n,\varphi,\psi\}$ which changes sign at $n+1$ given points in I, and it cannot be both of the form

$$g_1(t) = \varphi(t) - \sum_{i=1}^{n} a_i u_i(t), \quad \text{and}$$

$$g_2(t) = \psi(t) - \sum_{i=1}^{n} b_i u_i(t) .$$

Lemma 4.4. Let $h_1(t)$ and $h_2(t)$ be as in (4.2). Then for each $i = 1, \ldots, k-1$, there exists an $\eta_j \in (\xi_i, \xi_{i+1})$.

Proof. Assume that this is not the case. Replace $h_2(t)$ by $-h_2(t)$, if necessary, in order that $h_1(t) - h_2(t) \equiv 0$ for $t \in (\xi_i, \xi_{i+1})$. If $i = 1$, then $h_1(t) - h_2(t)$ has no sign change in $(0, \xi_3)$, while $S^-_{(\xi_3, 1)}(h_1 - h_2) \leq k-3$ by Lemma 4.2. Thus $S^-_{(0,1)}(h_1-h_2) \leq k-2 \leq n-1$, contradicting Lemma 4.1. The analogous result holds for $i = k-1$. Assume $1 < i < k-1$. Then $h_1(t) - h_2(t)$ has no sign change on (ξ_{i-1}, ξ_{i+2}), while $S^-_{(0, \xi_{i-1})}(h_1-h_2) \leq i-2$, and $S^-_{(\xi_{i+2}, 1)}(h_1-h_2) \leq k-i-2$. Therefore, $S^-_{(0,1)}(h_1-h_2) \leq (i-2) + (k-i-2) + 2 = k-2 \leq n-1$, a contradiction. The lemma is proven.

Proof of Theorem 4.1. If $S^-(g_1) = S^-(g_2) = n$, the result follows from Lemma 4.3. Assume this is not the case. Then Lemma 4.4 immediately implies that the zeros of $g_1(t)$ and $g_2(t)$ in $(0,1)$ strictly interlace. If $I = [0,1)$, and $g_1(0) = 0$, then $S^-(g_1) = n$ since $g_1(t)$ has at most $n+1$ zeros on I, and thus $S^-(g_2) = n+1$. The strict interlacing on I now follows. The same reasoning applies if $I = (0,1]$ or $I = [0,1]$, and the theorem is proven.

A scrutiny of the proof of Theorem 4.1 reveals that the Tchebycheffian property of $\{u_1, \ldots, u_n, \varphi, \psi\}$ has not been used except to establish a bound on the number of sign changes of $E_1(\varphi)$ and $E_1(\psi)$. Hence the same proof establishes the following.

Theorem 4.2. Let $\{u_i\}_{i=1}^n$ be a continuous T-system on I, continuous on $\bar{I}$, and let φ and ψ be linearly independent continuous functions on I

such that $E_1(\varphi)$ and $E_1(\psi)$ vanish on sets of measure 0 and change sign at no more than $n+1$ points in I. Then either the two sequences of points of sign change strictly interlace, or $\operatorname{sgn} E_1(\varphi)(t) = \operatorname{sgn} E_1(\psi)(t)$ for all $t \in \operatorname{int}(I)$.

5. $p = \infty$

The results for $p = \infty$ parallel those obtained for $p \in [1, \infty)$. Moreover, the proofs, in this case, involve no more than a careful zero counting procedure (cf. [3, Chap. 2]). Accordingly, we state the results without proof. Note that in this case we assume, in order that the best approximation be unique, that I is closed. Let $g_1(t)$ and $g_2(t)$ denote the error function in best L^∞ (Tchebycheff) approximation to $\varphi(t)$ and $\psi(t)$, respectively, from $[u_i(t)]_{i=1}^n$ on I. We assume that $\{u_i\}_{i=1}^n$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ are both T-systems on I. Then,

Theorem 5.1. The zeros of $g_1(t)$ and $g_2(t)$ in I strictly interlace.

From the characterization of best L^∞-approximation, it is known that $g_1(t)$ alternates at k points, $k = n+1$ or $n+2$, between $\|g_1\|_\infty$ and $-\|g_1\|_\infty$, and this same property obtains for $g_2(t)$. The fact that $\{u_1, \ldots, u_n, \varphi, \psi\}$ is a T-system on I immediately implies that $g_1(t)$ and $g_2(t)$ cannot both exhibit $n+2$ points of alternation on I. Furthermore, the following result is also obtainable by zero counting.

Theorem 5.2. The points of alternation of $g_1(t)$ and $g_2(t)$ on I weakly interlace.

Remark 5.1. If $\{u_1, \ldots, u_n, \varphi, \psi\}$ is an Extended Tchebycheff system of

order 2, in which case the zero counting is modified (see Karlin and Studden [3, Chap. 1]), we have strict interlacing in Theorem 5.2 in the interior of I.

Properties of the error function of the type exhibited in this section were established under more restrictive conditions by Rowland [6], Shohat [7], and Paszkowski [5].

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Assume $\{u_i\}_{i=1}^n$ and $\{u_1, \ldots, u_n, \varphi, \psi\}$ are T-systems on I, a finite interval. Let $E_p(\varphi)$ and $E_p(\psi)$ denote the error functions in the best L^p-approximation to φ and ψ, respectively, from $[u_1, \ldots, u_n]$. We show that for $1 < p < \infty$, the zeros of $E_p(\varphi)$ and $E_p(\psi)$ strictly interlace. This same result holds for $p = 1$ and $p = \infty$, with minor modifications.

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